Pre-class Warm-up!!!

What are the roots of the polynomial

$r^4 - 6r^2 + 9 = (r^2)^2 - 6(r^2) + 9$

$= (r^2 - 3)^2$

a. 3, -3

b. 3i, -3i

c. √3, -√3

d. √3, √3, -√3, -√3

e. None of the above.

$r^2 = \frac{6 \pm \sqrt{36 - 36}}{2} = 3$

$r = \pm\sqrt{3}$

5.3 Homogeneous equations with constant coefficients

We learn:

- More about the characteristic equations for linear d.e.'s with constant coefficients:
- The derivative as a linear operator on spaces of functions
- Repeated roots again
- Complex roots
- Euler's formula $e^{it} = \cos t + i \sin t$

Repeated roots of the characteristic equation

for a linear d.e. with constant coefficients

Review of what to do:

If the characteristic equation has a root λ repeated k times, the homogeneous equation has solutions $e^{\lambda x}, xe^{\lambda x}, \ldots, x^{k-1}e^{\lambda x}$

Example (like questions 1-20)

Find the general solution to $y^{(4)} - 6y'' + 9y = 0$

The characteristic equation $r^4 - 6r^2 + 9 = 0$ has roots $\sqrt{3}$, $-\sqrt{3}$, occurring twice.

The functions $e^{\sqrt{3}x}, xe^{\sqrt{3}x}, e^{-\sqrt{3}x}, xe^{-\sqrt{3}x}$ are basis for the solution space.

Why does this work?

The differential operator $D = \frac{d}{dx}$

$D: \{\text{functions } \mathbb{R} \to \mathbb{R}\} \to \{\text{functions } \mathbb{R} \to \mathbb{R}\}$.

has $D(af + bg) = aDf + bDg$. We can also add copies of D, and multiply D by itself

Write the differential equation on the left as

$(D^4 - 6D^2 + 9)y = 0$. This is the same as $(D^2-3)^2 y = 0 = (D-\sqrt{3})^2(D+\sqrt{3})^2 y$

Any solution of $(D-\sqrt{3})^2 y = 0$ is a solution of the d.e.

Compute (D+√3) e^{-√3 x}

$= De^{-\sqrt{3}x} + \sqrt{3}e^{-\sqrt{3}x}$

$= -\sqrt{3}e^{-\sqrt{3}x} + \sqrt{3}e^{-\sqrt{3}x} = 0$.

We just did (D+√3) e^{-√3 x} = 0

Next: (D+√3) x e^{-√3 x}

$$= D(xe^{-\sqrt{3}x}) + \sqrt{3}\, x e^{-\sqrt{3}x}$$

$$= e^{-\sqrt{3}x} - \sqrt{3}\, x e^{-\sqrt{3}x} + \sqrt{3}\, x e^{-\sqrt{3}x}$$

$$= e^{-\sqrt{3}x}$$

Thus $(D+\sqrt{3})^2 (x e^{-\sqrt{3}x}) = 0$

and $e^{-\sqrt{3}x}$, $x e^{-\sqrt{3}x}$ are a basis of solutions to $(D+\sqrt{3})^2 y = 0$.

$\sqrt{3}$ could have been any number λ.

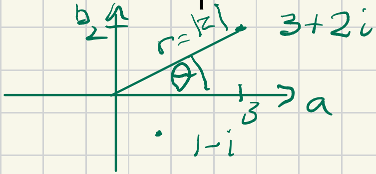
An independent set of solutions to $(D-\lambda)^k y = 0$ is:

$$e^{\lambda x},\ x e^{\lambda x},\ \ldots,\ x^{k-1} e^{\lambda x}.$$

Review of complex numbers

$i^2 = -1$

We may represent complex numbers $z = a + ib$ as points in the plane



The modulus or absolute value of z is $|z| = \sqrt{a^2 + b^2}$

The argument of z is $\theta = \tan^{-1} \frac{b}{a}$

Polar form: we can write $a + ib = r(\cos\theta + i\sin\theta)$

$$= re^{i\theta}$$

Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$

We can deduce Euler's formula if we know power series expansions of e^x, $\sin x$, $\cos x$.

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots$$

$$= 1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \cdots + i\theta + \frac{(i\theta)^3}{3!} + \cdots$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta + i\sin\theta.$$

The complex conjugate of $z = a + ib$ is

$$\bar{z} = a - ib$$

Note $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

Roots of polynomials with real coefficients occur in complex conjugate pairs

If $a+ib$ is a root of $x^n + a_{n-1}x^{n-1} + \cdots$ with a_i real, then $\overline{a+ib}$ is a root of the complex conjugate polynomial = same polynomial.

Questions

Let $z = -1 + i$

1. What is $|z|$?

a. $-\sqrt{2}$

b. $1/2$

c. $\sqrt{2}$

d. 2

e. None of the above.

2. What is $\arg(z)$?

a. $\pi/4$

b. $\pi/2$

c. $3\pi/4$

d. $2\pi/3$

e. None of the above.

Question like 5.3, 27-36

Find the general solution to $y^{(3)} - y'' + 2y = 0$
given that this equation is
(D+1)(D^2 - 2D + 2)y = 0 $= (D^3 - D^2 + 2)y$

Solution. The characteristic polynomial is

$(r+1)(r^2 - 2r+2)$ which has roots

$r = -1, \quad \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

The solutions $y = e^{-x}, \ e^{(1+i)x}, \ e^{(1-i)x}$

are a basis for the solution space.

Here $y = e^{(1+i)x} = e^x e^{ix} = e^x(\cos x + i \sin x)$

$y = e^{(1-i)x} = e^x(\cos x - i \sin x)$

We want a basis consisting of 3 real functions. We use the functions

$y = \frac{1}{2}\left(e^{(1+i)x} + e^{(1-i)x}\right) = e^x \cos x$

$y = \frac{1}{2i}\left(e^{(1+i)x} - e^{(1-i)x}\right) = e^x \sin x$

The solutions $e^{-x}, \ e^x \cos x, \ e^x \sin x$

are independent and form a basis for solutions.

The general solution is

$y = Ae^{-x} + Be^x \cos x + Ce^x \sin x$

For a complex root $z = a + ib$ of the characteristic equation we get a solution

$$e^{(a+ib)x} = e^{ax}(\cos bx + i\sin bx)$$

The equation probably has real coefficients and complex roots occur in complex conjugate pairs, so

$$e^{(a-ib)x} = e^{ax}(\cos bx - i\sin bx)$$

is also a solution. The two solutions

$$e^{ax}\cos bx \quad \text{and} \quad e^{ax}\sin bx$$

are also independent solutions spanning the same space.

$$\frac{e^{(a+ib)x} + e^{(a-ib)x}}{2} \qquad \frac{e^{(a+ib)x} - e^{(a-ib)x}}{2i}$$

Pre-class Warm-up!!!

What do you think the general solution is of the following equation:

$$(D^2 - 2D + 2)\, y = 0$$

Note that r^2 - 2r + 2 has roots 1 + i and 1 - i.

a. $a e^{x} \cos x + b e^{x} \sin x$ ✓

b. $a e^{x} \cos x + b x e^{x} \cos x + c e^{x} \sin x + d x e^{x} \sin x$

c. $a e^{-x} \cos x + b e^{-x} \sin x$

d. None of the above.

Question:
What do you think the general solution is of the following equation:

$$(D^2 - 2D + 2)^2 y = 0$$

Note that r^2 - 2r + 2 has roots 1 + i and 1 - i.

a. $a e^x \cos x + b e^x \sin x$

b. $a e^x \cos x + b x e^x \cos x + c e^x \sin x + d x e^x \sin x$

c. $a e^x \cos^2 x + b x e^x \cos^2 x + c e^x \sin^2 x + d x e^x \sin^2 x$

d. None of the above.

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One solution of the d.e. is given. Find the general solution.

9y^(3) + 11y″ + 4y′ - 14y = 0,   y = e^{-x} sin x

Solution: The characteristic polynomial is $9r^3+11r^2+4r-14$

The fact that $y=e^{-x}\sin x$ is a solution means that $-1+i$ and $-1-i$ are roots

so $(r-(-1+i))(r-(-1-i)) = r^2+2r+2$

because $(-1+i)(-1-i) = 1+i-i-i^2 = 2$

and $9r^3+11r^2+4r-14 = (r^2+2r+2)(9r-7)$

New root: $r = \frac{7}{9}$

The general solution is

$y = Ae^{7/9x} + Be^{-x}\cos x + Ce^{-x}\sin x$.

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Find a linear homogeneous constant-coefficient equation with the given general solution.

y(x) = (A + Bx + Cx^2) cos 2x + (D + Ex + Fx^2) sin 2x.

Solution. The characteristic polynomial has roots $2i, -2i$ repeated 3 times.

The polynomial is $(r-2i)^3(r+2i)^3$

$= (r^2+4)^3$.

The d.e. is $(D^2+4)^3 y = 0$